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Euler-*q* Algorithm for Attitude Determination from Vector Observations

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A new cost function for optimal attitude definition and the Euler-*q* algorithm based on this cost function are presented. The optimality criterion is derived from the Euler axis rotational property and allows a fast and reliable computation of the optimal eigenaxis. The mathematical procedure leads to the eigenanalysis of a 3 × 3 symmetric matrix whose eigenvector, associated with the smallest eigenvalue, is the optimal Euler axis. This eigenvector is evaluated by a simple cross vector, and the singularity is avoided using the method of sequential rotations. The rotational error is then analyzed and defined, and an accuracy comparison test is performed between a previously accepted criterion of optimal attitude and the proposed one. Results show that the earlier definition of optimality is slightly more precise than Euler-*q*, which, in turn, demonstrates a clear gain in computational speed.

Nomenclature

$\boldsymbol{A}, \boldsymbol{T}$	= estimation and true attitude matrix (direction-cosine matrix)
$\boldsymbol{B}, z$	= attitude data matrix and vector
$\boldsymbol{e}, \Phi$	= Euler axis (eigenaxis, principal axis) and Euler angle (principal angle)
L, G	= loss and gain functions, respectively
$\boldsymbol{S}$	= true direction of the observed $\boldsymbol{s}$ (unit vector)
$\boldsymbol{s}, \boldsymbol{v}$	= observed and referenced directions (unit vector)
α	= relative precision of attitude sensor
β	= precision of attitude sensor, deg
β^*	= precision of ($\boldsymbol{v}$–$\boldsymbol{s}$) direction, deg
λ	= eigenvalue, Lagrangian multiplier
ξ	= relative precision of ($\boldsymbol{v}$–$\boldsymbol{s}$) direction

Introduction

THE estimation of the orientation of a spacecraft body system (SBS) with respect to another reference system such as an inertial reference system (IRS), that is, the estimation of the spacecraft's attitude, is an important task of spacecraft navigation, dynamics, and control problems. This task, which has to be executed in the fastest and most precise manner, is often accomplished using the information of the n unit vectors $\boldsymbol{s}_i$, observed by the attitude sensors in the SBS, and the same directions evaluated by proper codes in the IRS, that is, the $\boldsymbol{v}_i$ unit vectors.

In the SBS the unit vector $\boldsymbol{v}_i$ is represented by the unit vector $\boldsymbol{A}\boldsymbol{v}_i$, where $\boldsymbol{A}$ is the unknown attitude matrix (to be computed) estimating the spacecraft orientation, which can be expressed in terms of the Euler axis $\boldsymbol{e}$ and the Euler angle Φ by

$$\boldsymbol{A} = \boldsymbol{I}\cos\Phi + (1 - \cos\Phi)\boldsymbol{e}\boldsymbol{e}^T - \tilde{\boldsymbol{e}}\sin\Phi \tag{1}$$

where $\boldsymbol{I}$ and $\tilde{\boldsymbol{e}}$ represent, respectively, the 3 × 3 identity matrix and the 3 × 3 skew-symmetric matrix performing the cross product. Let β_i be the precision of the ith sensor. This means that the true direction $\boldsymbol{S}_i$ can be displaced from the observed $\boldsymbol{s}_i$ by an angle smaller than β_i. The sensor relative precisions α_i are derived from the β_i as follows:

$$\alpha_i = \frac{1}{\beta_i \sum_{k=1}^{n}(1/\beta_k)} \tag{2}$$

As is easily observed, the relative precisions α_i are positive, less than the unit, and greater to the extent to which the sensors are more accurate and satisfy the relative condition

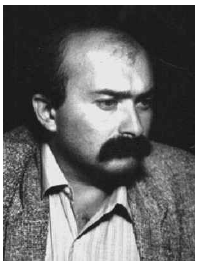
$$\sum_{i=1}^{n} \alpha_i$$

The optimal attitude estimation problem, which is known as the Wahba problem, means to estimate the spacecraft attitude by satisfying an optimality criterion and using the $n \geq 2$ unit vectors pairs $(\boldsymbol{s}_i, \boldsymbol{v}_i)$. Up to now the optimality criterion,[1] originally introduced by Wahba in 1965 and then completed by including the weights α_i as well as the $\frac{1}{2}$ multiplier, defines as optimal the attitude matrix $\boldsymbol{A}$ that minimizes the loss function

$$L_W(\boldsymbol{A}) = \frac{1}{2}\sum_{i=1}^{n}\alpha_i \|\boldsymbol{s}_i - \boldsymbol{A}\boldsymbol{v}_i\|^2 = 1 - \sum_{i=1}^{n}\alpha_i \boldsymbol{s}_i^T \boldsymbol{A}\boldsymbol{v}_i \tag{3}$$

Equivalently, optimality is reached when the gain function

$$G_W(\boldsymbol{A}) = 1 - L_W(\boldsymbol{A}) = \sum_{i=1}^{n}\alpha_i \boldsymbol{s}_i^T \boldsymbol{A}\boldsymbol{v}_i = \mathrm{tr}\,[\boldsymbol{A}\boldsymbol{B}^T] \tag{4}$$



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is maximal, where $\boldsymbol{B}$ has the form

$$\boldsymbol{B} = \sum_{i=1}^{n} \alpha_i \boldsymbol{s}_i \boldsymbol{v}_i^T \tag{5}$$

Now, substituting for the attitude matrix its expression given by Eq. (1), the general expression of the Wahba gain function expressed in terms of the Euler axis and angle becomes

$$G_W(\boldsymbol{e}, \Phi) = \cos\Phi \,\mathrm{tr}\,[\boldsymbol{B}] + (1 - \cos\Phi)\boldsymbol{e}^T\boldsymbol{B}\boldsymbol{e} + \sin\Phi \boldsymbol{z}^T\boldsymbol{e} \tag{6}$$

where the vector

$$\boldsymbol{z} = \sum_i \alpha_i \boldsymbol{s}_i \times \boldsymbol{v}_i$$

which indicates the nonsymmetricity of $\boldsymbol{B}$, can be directly expressed using the off-diagonal elements of $\boldsymbol{B}$ as $\boldsymbol{z} = \{b_{23} - b_{32}, b_{31} - b_{13}, b_{12} - b_{21}\}^T$. All of the existing algorithms, which fully comply with Wahba's optimality criterion of Eqs. (3) and (4), achieve the same accuracy. Thus, those algorithms, e.g., $\boldsymbol{q}$-Method,[2] QUEST,[3] SVD,[4] Direct-Method,[5] EAA,[6] Euler-2, TRIAD-2, and Euler-n (Ref. 7), differ from each other only in their computational speed. Particular attention was therefore given to the development of faster attitude determination techniques, mainly because an increase of the computational speed allows the attitude information to be provided to the control system at a higher rate. Reference 7 analyzes the possibility of computing separately the optimal Euler axis and angle ($\boldsymbol{e}_{\text{opt}}$, Φ_{opt}) using the Wahba optimal definition. This led to the development of the Euler-2 (when $n = 2$) and Euler-n (when $n > 2$) algorithms. Euler-n evaluates the optimal Euler axis $\boldsymbol{e}_{\text{opt}}$ and the optimal Euler angle Φ_{opt} in two consecutive steps of an iterative procedure. However, it turns out that it is possible to compute separately $\boldsymbol{e}_{\text{opt}}$ and Φ_{opt} when $n > 2$, without using iterative procedures, but this is achieved when a different criterion for optimality of the attitude determination is used, as shown in the next section.

Euler-q Algorithm

As already stated, in the ideal case of no noise ($\beta_i = 0, i = 1 - n$), the attitude matrix satisfies all of the $\boldsymbol{A}\boldsymbol{v}_i = \boldsymbol{s}_i$ conditions. Premultiplying $\boldsymbol{A}\boldsymbol{v}_i = \boldsymbol{s}_i$ by $\boldsymbol{e}^T$, with $\boldsymbol{A}$ given by Eq. (1), the identities

$$\boldsymbol{e}^T\boldsymbol{v}_i = \boldsymbol{e}^T\boldsymbol{s}_i \tag{7}$$

are derived. Note that Eq. (7) is a consequence of the attitude matrix being a rotational operator, which rotates, about the axis $\boldsymbol{e}$ by the angle $-\Phi$, the overall structure of the referenced vectors $\boldsymbol{v}_i$ until it overlaps with the structure of the observed vectors $\boldsymbol{s}_i$. However, due to errors, in the real case the conditions $\boldsymbol{e}^T\boldsymbol{v}_i = \boldsymbol{e}^T\boldsymbol{s}_i$, which imply $\boldsymbol{e}^T(\boldsymbol{v}_i - \boldsymbol{s}_i) = (\boldsymbol{v}_i - \boldsymbol{s}_i)^T\boldsymbol{e} = 0$, do not hold anymore; therefore, introducing the unit vectors $\boldsymbol{d}_i = (\boldsymbol{v}_i - \boldsymbol{s}_i)/\|\boldsymbol{v}_i - \boldsymbol{s}_i\|$, the small differences

$$\delta_i = \boldsymbol{e}^T\boldsymbol{d}_i = \boldsymbol{d}_i^T\boldsymbol{e} \tag{8}$$

arise in the real case. Let us define the loss function

$$L_M(\boldsymbol{e}) = \sum_{i=1}^{n} \xi_i\delta_i^2 = \sum_{i=1}^{n} \xi_i\boldsymbol{e}^T\boldsymbol{d}_i\boldsymbol{d}_i^T\boldsymbol{e} = \boldsymbol{e}^T\boldsymbol{H}\boldsymbol{e} \tag{9}$$

where the symmetric matrix

$$\boldsymbol{H} = \sum_{i=1}^{n} \xi_i\boldsymbol{d}_i\boldsymbol{d}_i^T \tag{10}$$

is introduced and the relative weights ξ_i are defined as follows. The more the $\boldsymbol{d}_i$ direction is perpendicular to the Euler axis, the more the condition given by Eq. (7) is satisfied. The deviation of $\boldsymbol{d}_i$ from the perpendicular condition depends on the sensor errors β_i; therefore, the relative weight ξ_i, establishing the $\boldsymbol{d}_i$ relative precision, is a function of the β_i angle. Referring to the spherical triangle identified by the directions $\boldsymbol{v}_i$, $\boldsymbol{s}_i$, and $\boldsymbol{S}_i$, shown in Fig. 1, the worst case for the $\boldsymbol{d}_i$ direction deviation (that is, the β_i^* angle) occurs when the true observed direction $\boldsymbol{S}_i$ is displaced from the measured $\boldsymbol{s}_i$ by the angle β_i and the spherical triangle is right in $\boldsymbol{S}_i$. In this position

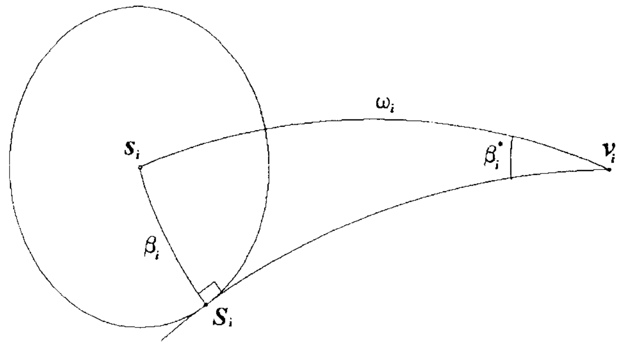

Fig. 1 Error in the $(\boldsymbol{v}_i - \boldsymbol{s}_i)$ direction.

the β_i^* angle, which represents the maximum deviation of $\boldsymbol{d}_i$, is maximized and its value is obtained from

$$\sin\omega_i \sin\beta_i^* = \sin\beta_i \tag{11}$$

where ω_i is the angle between the $\boldsymbol{s}_i$ and $\boldsymbol{v}_i$ directions. The relative weights ξ_i are then derived from the β_i^* as the α_i are obtained from the β_i, that is,

$$\xi_i = \frac{1}{\beta_i^* \sum_{k=1}^{n} \left(1/\beta_k^*\right)} \tag{12}$$

The minimization of $L_M(\boldsymbol{e})$ can be obtained by means of the Lagrangian multiplier technique. To this end, the augmented cost function, which includes the constraint of the Euler axis to be a unit vector, is defined as

$$L_M^*(\boldsymbol{e}) = \boldsymbol{e}^T\boldsymbol{H}\boldsymbol{e} - \lambda(\boldsymbol{e}^T\boldsymbol{e} - 1) \tag{13}$$

The stationarity condition implies

$$\frac{\mathrm{d}L_M^*(\boldsymbol{e})}{\mathrm{d}\boldsymbol{e}} = 2\boldsymbol{H}\boldsymbol{e} - 2\lambda\boldsymbol{e} = \boldsymbol{0} \tag{14}$$

which leads to

$$\boldsymbol{H}\boldsymbol{e} = \lambda\boldsymbol{e} \tag{15}$$

Equation (15) states that the searched Euler axis is the eigenvector of the $\boldsymbol{H}$ matrix associated with the λ eigenvalue. This eigenvalue is computed by premultiplying Eq. (15) by $\boldsymbol{e}^T$; thus,

$$\boldsymbol{e}^T\boldsymbol{H}\boldsymbol{e} = \lambda = L_M(\boldsymbol{e}) \tag{16}$$

Now, because $L_M(\boldsymbol{e})$ has to be a minimum, the unknown λ eigenvalue has to be the smallest. Therefore, the optimal Euler axis $\boldsymbol{e}_{\text{opt}}$ is the $\boldsymbol{H}$ matrix eigenvector associated with its smallest eigenvalue:

$$\boldsymbol{H}\boldsymbol{e}_{\text{opt}} = \lambda_{\min}\boldsymbol{e}_{\text{opt}} \tag{17}$$

Although Eq. (17) provides the solution to the problem, the complete eigenanalysis is unnecessary, as is demonstrated hereafter. In fact, it is possible to compute $\lambda_{\min}$ directly from the third-order characteristic equation of the $\boldsymbol{H}$ matrix

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \tag{18}$$

where the a, b, and c coefficients can be expressed in terms of the $\boldsymbol{H}$ matrix elements as

$$\begin{aligned} a &= -\,\mathrm{tr}[\boldsymbol{H}] = -h_{11} - h_{22} - h_{33} \\ b &= \mathrm{tr}[\mathrm{adj}(\boldsymbol{H})] = h_{11}h_{22} + h_{11}h_{33} + h_{22}h_{33} - h_{12}^2 - h_{13}^2 - h_{23}^2 \\ c &= -\det(\boldsymbol{H}) = h_{11}h_{22}h_{33} + 2h_{12}h_{13}h_{23} \\ &\quad - h_{22}h_{13}^2 - h_{11}h_{23}^2 - h_{33}h_{12}^2 \end{aligned} \tag{19}$$

Because $\boldsymbol{H}$ is symmetric, its eigenvalues are real; therefore, the solution of the third-order Eq. (18), providing three real roots, has to be chosen. This is accomplished[8,9] by setting

$$p^2 = (a/3)^2 - (b/3), \qquad q = [(b/2) - (a/3)^2](a/3) - (c/2) \tag{20}$$

and

$$w = \tfrac{1}{3}\cos^{-1}(q/p^3) \tag{21}$$

Then, the condition of having real roots leads to the solution

$$\begin{aligned} \lambda_1 &= -p\left(\sqrt{3}\sin w + \cos w\right) - a/3 \\ \lambda_2 &= p\left(\sqrt{3}\sin w - \cos w\right) - a/3 \\ \lambda_3 &= 2p\cos w - a/3 \end{aligned} \tag{22}$$

Because $0 \le w \le \pi/3$, it is easy to demonstrate that the various λ_i computed by Eq. (22) satisfy the conditions

$$0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \tag{23}$$

When $n = 2$, the coefficient $c = -\det(\boldsymbol{H}) = 0$. In this case the three roots of Eq. (18) become

$$\lambda_1 = 0 \qquad \text{and} \qquad \begin{cases} \lambda_2 = -a/2 - \sqrt{(a/2)^2 - b} \\ \lambda_3 = -a/2 + \sqrt{(a/2)^2 - b} \end{cases} \tag{24}$$

which still satisfy the conditions of Eq. (23). Hence, when $n = 2$, the computation of $\lambda_{\min}$ is unnecessary because it results in $\lambda_{\min} = \lambda_1 = 0$.

Obviously, because $\lambda_{\min}$ is very small [$\lambda_{\min} = L_M(\boldsymbol{e}) \simeq 0$; see Eq. (16)], it also can be computed in a very fast way by applying the Newton–Raphson iterative procedure to Eq. (18) with $\lambda_0 = 0$ as starting point. The first iteration implies $\lambda_1 = -c/b$, whereas the subsequent values can be obtained recursively by

$$\lambda_{i+1} = \lambda_i - \frac{\lambda_i^3 + a\lambda_i^2 + b\lambda_i + c}{3\lambda_i^2 + 2a\lambda_i + b} \tag{25}$$

The iterative procedure converges to the solution so dramatically that only one application of Eq. (25) implies a solution approximated to 10^{-15}. This iterative procedure has, with respect to the closed-form solution, the double advantage of requiring fewer operations (that is, a gain in computational speed) and avoiding the computation of square roots and inverse cosine functions [see Eqs. (20–22)], which reduce the numerical errors, because typical spaceborne microprocessors have limited word length. However, the closed-form solution is still suggested because it allows the computation of $(\lambda_2 - \lambda_{\min})$ and $(\lambda_3 - \lambda_{\min})$, which, in turn, indicate the solution robustness as well as the distance to the singularity.

Once $\lambda_{\min}$ is computed, $\boldsymbol{e}_{\text{opt}}$ is evaluated in an easy and fast manner. To this end, let us write Eq. (17) as follows:

$$(\boldsymbol{H} - \lambda_{\min}\boldsymbol{I})\boldsymbol{e}_{\text{opt}} = \boldsymbol{M}\boldsymbol{e}_{\text{opt}} = \begin{bmatrix} \boldsymbol{m}_1^T \\ \boldsymbol{m}_2^T \\ \boldsymbol{m}_3^T \end{bmatrix} \boldsymbol{e}_{\text{opt}}$$

$$= \begin{bmatrix} m_a & m_x & m_y \\ m_x & m_b & m_z \\ m_y & m_z & m_c \end{bmatrix} \boldsymbol{e}_{\text{opt}} = \boldsymbol{0} \tag{26}$$

This equation states that all of the row vectors of $\boldsymbol{M}$ are perpendicular to $\boldsymbol{e}$. Therefore, the direction of the optimal Euler axis can be computed by a simple cross product between two row vectors of the $\boldsymbol{M}$ matrix.

To maximize the accuracy, the computation of the optimal Euler axis by means of a cross product leads to the problem of which row vectors should be selected for computing it. To this end the following three choices are available:

$$\begin{aligned} \boldsymbol{e}_1 &= \boldsymbol{m}_2 \times \boldsymbol{m}_3 = \left\{m_b m_c - m_z^2,\, m_y m_z - m_x m_c,\, m_x m_z - m_y m_b\right\}^T \\ \boldsymbol{e}_2 &= \boldsymbol{m}_3 \times \boldsymbol{m}_1 = \left\{m_y m_z - m_x m_c,\, m_a m_c - m_y^2,\, m_x m_y - m_z m_a\right\}^T \\ \boldsymbol{e}_3 &= \boldsymbol{m}_1 \times \boldsymbol{m}_2 = \left\{m_x m_z - m_y m_b,\, m_x m_y - m_z m_a,\, m_a m_b - m_x^2\right\}^T \end{aligned} \tag{27}$$

where all of the $\boldsymbol{e}_i$ are parallel and, therefore, differ from each other only in modulus. To evaluate the $\boldsymbol{e}_i$ with highest reliability, it is necessary to identify which one has the greatest modulus. It is easy to demonstrate that this requirement is satisfied by computing which of

$$p_1 = \left|m_b m_c - m_z^2\right|, \qquad p_2 = \left|m_a m_c - m_y^2\right|, \qquad p_3 = \left|m_a m_b - m_x^2\right| \tag{28}$$

is the greatest. Let p_k be the greatest; then the most reliable Euler axis is given as

$$\boldsymbol{e}_{\text{opt}} = \boldsymbol{e}_k / \|\boldsymbol{e}_k\| \tag{29}$$

The computation of the Euler angle can be executed by performing the derivative of Eq. (6) with respect to the Euler angle. This leads to

$$\frac{\partial G_W(\boldsymbol{e}_{\text{opt}}, \Phi)}{\partial \Phi} = \left(\boldsymbol{e}_{\text{opt}}^T \boldsymbol{B} \boldsymbol{e}_{\text{opt}} - \text{tr}[\boldsymbol{B}]\right)\sin\Phi + \boldsymbol{z}^T \boldsymbol{e}_{\text{opt}} \cos\Phi = 0 \tag{30}$$

Therefore, the optimal Euler angle can be derived from the following conditions:

$$\begin{aligned} \sin\Phi_{\text{opt}} &= (1/D)\boldsymbol{z}^T \boldsymbol{e}_{\text{opt}} \\ \cos\Phi_{\text{opt}} &= (1/D)\left(\text{tr}[\boldsymbol{B}] - \boldsymbol{e}_{\text{opt}}^T \boldsymbol{B} \boldsymbol{e}_{\text{opt}}\right) \end{aligned} \tag{31}$$

where

$$D^2 = \left(\boldsymbol{z}^T \boldsymbol{e}_{\text{opt}}\right)^2 + \left(\text{tr}[\boldsymbol{B}] - \boldsymbol{e}_{\text{opt}}^T \boldsymbol{B} \boldsymbol{e}_{\text{opt}}\right)^2 \tag{32}$$

Singularity Discussion

The computation of the Euler axis by means of a cross product fails in two limit cases for the $\boldsymbol{M}$ row vectors: 1) when they become parallel, which occurs either when all of the observed vectors are parallel or thereabouts (a-singularity), or when the Euler axis and all of the observed vectors are approximately lying on the same plane (b-singularity); and 2) when they become zero vectors, which occurs if $\Phi \to 0$ (c-singularity).

The a-singularity, which occurs when the observed vectors are all parallel, implies that the $\boldsymbol{d}_i$ vectors are parallel as well, and therefore the matrix $\boldsymbol{H}$ will result as if it were built using only one vector. Such a matrix has nonnegative eigenvalues, which are the square roots of the singular values. Therefore, this matrix has rank 1, and thus $\lambda_3 > 0$, whereas $\lambda_2 \to \lambda_1 \to 0$. This singularity case cannot be solved (even by using other existing algorithms) because it is an intrinsically unresolvable case for attitude determination.

The b-singularity also implies that the $\boldsymbol{d}_i$ vectors are parallel, and therefore the same consequences arise as for the a-singularity, which are $\lambda_3 > 0$, whereas $\lambda_2 \to \lambda_1 \to 0$. Unlike the a-singularity, the b-singularity can be solved by using additional vectors, as explained in the next section.

Finally, the c-singularity, which occurs for $\Phi \to 0$, implies that first $\boldsymbol{A} \to \boldsymbol{I}$, then $\boldsymbol{s}_i \to \boldsymbol{v}_i$, then $(\boldsymbol{v}_i - \boldsymbol{s}_i) \to \boldsymbol{0}$, and finally $\boldsymbol{H} \to \boldsymbol{0}$, which in its turn implies that all of the eigenvalues of $\boldsymbol{H}$ are tending to zero, that is, $\lambda_3 \to \lambda_2 \to \lambda_1 \to 0$ and, consequently, $\boldsymbol{M} \to \boldsymbol{H} \to \boldsymbol{0}$, i.e., zero row vectors for matrix $\boldsymbol{M}$.

To avoid the b- and c-singularities, the method of sequential rotations (MSR)[3] can successfully be applied. MSR is a simple and elegant solution tool that can be invoked in almost all of the singular

Table 1 Sequential rotation properties

MSR axis	MSR angle	$\boldsymbol{w}$ vectors	$\boldsymbol{B}^*$ matrix	$\boldsymbol{A}$ matrix	$\boldsymbol{q}$ quaternion
$\boldsymbol{c}_1$	π	$\{\boldsymbol{v}(1)\ -\boldsymbol{v}(2)\ -\boldsymbol{v}(3)\}^T$	$[\boldsymbol{b}_1\ -\boldsymbol{b}_2\ -\boldsymbol{b}_3]$	$[f_1\ -f_2\ -f_3]$	$\{p_4\ -p_3\ p_2\ -p_1\}^T$
$\boldsymbol{c}_2$	π	$\{-\boldsymbol{v}(1)\ \boldsymbol{v}(2)\ -\boldsymbol{v}(3)\}^T$	$[-\boldsymbol{b}_1\ \boldsymbol{b}_2\ -\boldsymbol{b}_3]$	$[-f_1\ f_2\ -f_3]$	$\{p_3\ p_4\ -p_1\ -p_2\}^T$
$\boldsymbol{c}_3$	π	$\{-\boldsymbol{v}(1)\ -\boldsymbol{v}(2)\ \boldsymbol{v}(3)\}^T$	$[-\boldsymbol{b}_1\ -\boldsymbol{b}_2\ \boldsymbol{b}_3]$	$[-f_1\ -f_2\ f_3]$	$\{-p_2\ p_1\ p_4\ -p_3\}^T$

Table 2 Roots difference table

Eigenvalue difference	$n = 2$	$n > 2$
$\lambda_2 - \lambda_1$	$\left(-a - \sqrt{a^2 - 4b}\right)/2$	$2p\sqrt{3}\sin w$
$\lambda_3 - \lambda_1$	$\sqrt{a^2 - 4b}$	$p\left(3\cos w + \sqrt{3}\sin w\right)$
$\lambda_3 - \lambda_2$	$\left(-a + \sqrt{a^2 - 4b}\right)/2$	$p\left(3\cos w - \sqrt{3}\sin w\right)$

cases encountered in the attitude estimation algorithms subjected to a singularity. MSR states that, if the n unit vector pairs $(\boldsymbol{s}_i, \boldsymbol{v}_i)$ imply the optimal attitude matrix $\boldsymbol{A}$, then the n unit vector pairs $(\boldsymbol{s}_i, \boldsymbol{w}_i)$, where $\boldsymbol{w}_i = \boldsymbol{R}\boldsymbol{v}_i$ and $\boldsymbol{R}$ is any one rotational matrix, imply the optimal attitude matrix $\boldsymbol{F}$, related to $\boldsymbol{A}$, as $\boldsymbol{F} = [f_1\ f_2\ f_3] = \boldsymbol{A}\boldsymbol{R}^T$. Therefore, if the $(\boldsymbol{s}_i, \boldsymbol{v}_i)$ data set implies a singularity for the application method, the $(\boldsymbol{s}_i, \boldsymbol{w}_i)$ set would not, in general, necessarily imply a singularity too. Hence, the MSR evaluates the attitude $\boldsymbol{F}$ (or the quaternion $\boldsymbol{p}$) by using the rotated unit vectors $\boldsymbol{w}_i$ (in place of the $\boldsymbol{v}_i$) and then computes the searched optimal attitude matrix as $\boldsymbol{A} = \boldsymbol{F}\boldsymbol{R}$. Particular attention is given to those matrices $\boldsymbol{R}$ that rotate about one of the coordinate axes $\boldsymbol{c}_1$, $\boldsymbol{c}_2$, and $\boldsymbol{c}_3$ by a π angle. These matrices, for which the rules given in Table 1 are valid, are

$$\boldsymbol{R}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \boldsymbol{R}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
$$\boldsymbol{R}_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{33}$$

With respect to other matrices, these save time in computations. In fact, the application of the MSR with them does not involve extra computational loads but sign changes only. When applying the MSR by a π rotation about one of the coordinate axes, the computation of the new data matrix $\boldsymbol{B}^*$ is unnecessary because $\boldsymbol{B}^*$ is obtained from $\boldsymbol{B} = [\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3]$ just by changing the sign of two columns; the attitude matrix $\boldsymbol{A}$ is also obtained from $\boldsymbol{F}$ by sign changes; the optimal searched quaternion $\boldsymbol{q} = \{q_1\ q_2\ q_3\ q_4\}^T$, associated with $\boldsymbol{B}$, is also obtained from the computed quaternion $\boldsymbol{p} = \{p_1\ p_2\ p_3\ p_4\}^T$, associated with $\boldsymbol{B}^*$, by sign changes and cross displacements. All of these properties are summarized in Table 1.

The conditions under which the MSR is invoked are related to the values of the computed roots differences, the expressions of which are provided in Table 2. This represents another advantage of computing the λ_i by the closed-form solution instead of the iterative procedure.

The condition $\lambda_2 - \lambda_1 > \zeta$, where ζ is a small positive tolerance to be derived from numerical tests by limiting the attitude error, implies an optimal Euler axis sufficiently identifiable. In this case the MSR is not invoked. When $\lambda_2 - \lambda_1 < \zeta$, then the MSR is invoked about another axis until the condition $\lambda_2 - \lambda_1 > \zeta$ is satisfied. As is known, the condition $\lambda_3 - \lambda_1 < \zeta$ identifies the c-singularity that is solvable. Finally, when the condition $\lambda_2 - \lambda_1 > \zeta$ does not occur for all three possible sequential rotations, then it means that the a-singularity is occurring. When this case occurs, obviously, the attitude estimation is aborted.

Additional Vectors

The b-singularity implies that the $\boldsymbol{d}_i$ vectors are parallel, and therefore the same consequences arise as for the a-singularity, which are $\lambda_3 > 0$, whereas $\lambda_2 \to \lambda_1 \to 0$. It is possible, however, to avoid completely the b-singularity using the additional vectors, hereafter described.

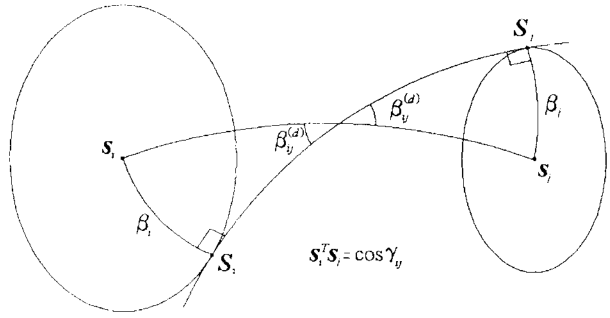

Fig. 2 Error in the $(\boldsymbol{s}_i \times \boldsymbol{s}_j)$ direction.

In fact, because the attitude matrix represents a rotational operator, not only does each single $\boldsymbol{v}_i$ describe a cone about $\boldsymbol{e}$ but also any other vector constructed using the $\boldsymbol{v}_i$, as, for instance, the cross products $\boldsymbol{v}_i \times \boldsymbol{v}_j$. This means that the $\boldsymbol{s}_i \times \boldsymbol{s}_j$ vector can be considered as an additional observed vector, whose corresponding reference vector is $\boldsymbol{v}_i \times \boldsymbol{v}_j$. Therefore, in the ideal case, the conditions $\boldsymbol{e}^T(\boldsymbol{s}_i \times \boldsymbol{s}_j) = \boldsymbol{e}^T(\boldsymbol{v}_i \times \boldsymbol{v}_j)$, which imply that the directions provided by the vectors $\boldsymbol{d}_{ij} = (\boldsymbol{v}_i \times \boldsymbol{v}_j - \boldsymbol{s}_i \times \boldsymbol{s}_j)/\|\boldsymbol{v}_i \times \boldsymbol{v}_j - \boldsymbol{s}_i \times \boldsymbol{s}_j\|$ are perpendicular to the Euler axis, are satisfied.

Consequently, the small differences

$$\xi_{ij}\delta_{ij}^2 = \xi_{ij}\boldsymbol{e}^T\boldsymbol{d}_{ij}\boldsymbol{d}_{ij}^T\boldsymbol{e} \tag{34}$$

where the ξ_{ij}, the expressions of which are more complicated than that of the ξ_i, are provided at the end of this section. Terms given in Eq. (34) can be added to the loss function $L_M(\boldsymbol{e})$ of Eq. (9). These additional vectors avoid completely the occurrence of the b-singularity. In fact, it happens that, when the directions $\boldsymbol{s}_i$, $\boldsymbol{s}_j$, and $\boldsymbol{e}$ are lying on the same plane, the vector $(\boldsymbol{v}_i \times \boldsymbol{v}_j - \boldsymbol{s}_i \times \boldsymbol{s}_j)$ is perpendicular to both the Euler axis and the parallel vectors $\boldsymbol{d}_i$ and $\boldsymbol{d}_j$.

When an additional vector is considered, the true $\boldsymbol{D}_{ij}$ direction is displaced from the observed $\boldsymbol{d}_{ij}$ because $\boldsymbol{S}_i \times \boldsymbol{S}_j$ differs from $\boldsymbol{s}_i \times \boldsymbol{s}_j$ in both direction and modulus. The variation in direction is derived considering the vector $\boldsymbol{S}_i \times \boldsymbol{S}_j$ describing a cone about $\boldsymbol{s}_i \times \boldsymbol{s}_j$ by the angle $\beta_{ij}^{(d)}$. The maximum value of this angle, which is reached in the condition shown in Fig. 2, can be computed by the relationship

$$\sin^2\gamma_{ij}\sin^2\beta_{ij}^{(d)} = \sin^2\beta_i + \sin^2\beta_j + 2\sin\beta_i\sin\beta_j\cos\gamma_{ij} \tag{35}$$

where γ_{ij} is the angle between $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$. Equation (35) is derived by applying known trigonometric identities to the spherical figure identified by the $\boldsymbol{s}_i$, $\boldsymbol{S}_i$, $\boldsymbol{s}_j$, and $\boldsymbol{S}_j$ directions. This means that the vector $\boldsymbol{s}_i \times \boldsymbol{s}_j$ is displaced from $\boldsymbol{S}_i \times \boldsymbol{S}_j$ by a maximum angle of $\beta_{ij}^{(d)}$, and therefore the $\boldsymbol{d}_{ij}$ direction can be displaced from $\boldsymbol{D}_{ij}$ by the angle $\beta_{ij}^{(d)*}$, which can be computed from the relationship

$$\sin\omega_{ij}\sin\beta_{ij}^{(d)*} = \sin\beta_{ij}^{(d)} \tag{36}$$

where ω_{ij} is the angle between the $\boldsymbol{s}_i \times \boldsymbol{s}_j$ and $\boldsymbol{v}_i \times \boldsymbol{v}_j$ directions.

The contribution caused by the modulus variation $\|\boldsymbol{s}_i \times \boldsymbol{s}_j\|$ can be considered approximately perpendicular to that caused by the variation in direction. Referring to the triangle shown in Fig. 3, it is possible to compute the error angle $\beta_{ij}^{(m)*}$ associated with the $\|\boldsymbol{s}_i \times \boldsymbol{s}_j)\|$ variation.

In Table 3, the minimum and the maximum values for $m^{(s)} = \|\boldsymbol{s}_i \times \boldsymbol{s}_j\| = \sin(\gamma_{ij} + \beta)$, where the variable β ranges from $-\beta_s$ to β_s (where $\beta_s = \beta_i + \beta_j$), are given as a function of the value of γ_{ij}.

Table 3 $\|s_i \times s_j\|$ **range**

γ_{ij} range		$m^{(s)} = \|s_i \times s_j\|$ range	
From	To	$m^{(s)}_{\max}$	$m^{(s)}_{\min}$
0	β_s	0	$\sin(\gamma_{ij} + \beta_s)$
β_s	$\pi/2 - \beta_s$	$\sin(\gamma_{ij} - \beta_s)$	$\sin(\gamma_{ij} + \beta_s)$
$\pi/2 - \beta_s$	$\pi/2$	$\sin(\gamma_{ij} - \beta_s)$	1
$\pi/2$	$\pi/2 + \beta_s$	$\sin(\gamma_{ij} + \beta_s)$	1
$\pi/2 + \beta_s$	$\pi - \beta_s$	$\sin(\gamma_{ij} + \beta_s)$	$\sin(\gamma_{ij} - \beta_s)$
$\pi - \beta_s$	π	0	$\sin(\gamma_{ij} - \beta_s)$

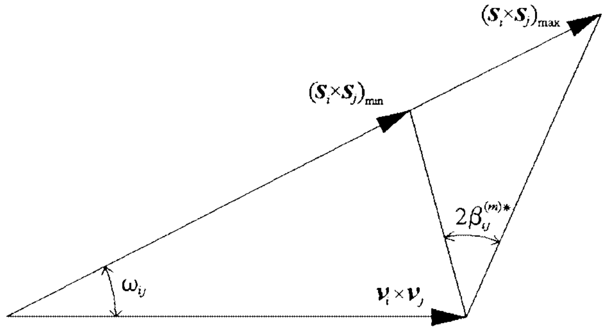

Fig. 3 Error in the $(s_i \times s_j)$ **modulus.**

The angle $2\beta_{ij}^{(m)*}$, which is the angle spanned by the vector $(\boldsymbol{v}_i \times \boldsymbol{v}_j - \boldsymbol{s}_i \times \boldsymbol{s}_j)$ during the β variation, can be evaluated from the equation

$$2z_1 z_2 \cos\left[2\beta_{ij}^{(m)*}\right] = z_1^2 + z_2^2 - \left[m^{(s)}_{\max} - m^{(s)}_{\min}\right]^2 \tag{37}$$

where

$$\begin{aligned} z_1^2 &= \left[m^{(s)}_{\max}\right]^2 + \|\boldsymbol{v}_i \times \boldsymbol{v}_j\|^2 - 2\|\boldsymbol{v}_i \times \boldsymbol{v}_j\| m^{(s)}_{\max} \cos\omega_{ij} \\ z_2^2 &= \left[m^{(s)}_{\min}\right]^2 + \|\boldsymbol{v}_i \times \boldsymbol{v}_j\|^2 - 2\|\boldsymbol{v}_i \times \boldsymbol{v}_j\| m^{(s)}_{\min} \cos\omega_{ij} \end{aligned} \tag{38}$$

Once the precision angles $\beta_{ij}^{(d)*}$ and $\beta_{ij}^{(m)*}$ are computed, then $\beta_{ij}^* = \max[\beta_{ij}^{(m)*}, \beta_{ij}^{(d)*}]$.

The relative weights ξ_{ij}, which appeared in Eq. (34), are then evaluated as the α_i were evaluated from the β_i in Eq. (2). For instance, when $n = 2$ and an additional vector is considered, the three weights are

$$\xi_1 = \frac{1}{D\beta_1^*}, \qquad \xi_2 = \frac{1}{D\beta_2^*}, \qquad \xi_3 = \frac{1}{D\beta_{12}^*} \tag{39}$$

where

$$D = 1/\beta_1^* + 1/\beta_2^* + 1/\beta_{12}^* \tag{40}$$

and where β_1^* and β_2^* are computed using Eq. (12), whereas β_{12}^* is evaluated using Eqs. (35–38). The use of the additional vectors can be restricted to when $n = 2$ only (which means only one additional vector) for the following three reasons: first, because the b-singularity is completely avoided using the MSR; second, because the b-singularity occurrence highly decreases with n; and third, because additional vectors imply additional computational load, especially for the relative weights ξ_{ij} computation. Note, however, that the use of the additional vectors implies a small improvement of the solution accuracy.

Rotation Error

Any criterion used to establish the optimality of the spacecraft attitude implies a corresponding optimal attitude matrix. Typically, this matrix does not coincide perfectly with the true attitude matrix describing the spacecraft orientation, and therefore any estimation of the spacecraft attitude is affected by an error that needs quantification. Because the attitude matrix is a rotational matrix, the problem is to identify which mathematical means is apt to measure the distance between two rotational matrices, that is, the error of an estimation with respect to the true attitude matrices.

To this end, let $\boldsymbol{T}$ and $\boldsymbol{A}$ be the true and the estimation attitude matrix, respectively. Matrix $\boldsymbol{A}$ can even be randomly chosen provided that the conditions of being an attitude matrix are met $[\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}, \det(\boldsymbol{A}) = 1]$. The problem is to evaluate numerically, in an unambiguous manner, how far $\boldsymbol{A}$ is from $\boldsymbol{T}$. To that end, in the past literature, different criteria have been introduced. One of them is based on the computation of the Euclidean (Frobenius) norm of the matrix $\delta\boldsymbol{A} = \boldsymbol{T} - \boldsymbol{A}$ (Ref. 5), and another introduces the attitude error vector,[10] which can be considered correct only for infinitesimal deviations between $\boldsymbol{A}$ and $\boldsymbol{T}$. Both of these criteria calculate quantities that, although related to the error, are not the error itself. In the following it is shown that the attitude error, that is, the rotational error, has an assigned shape that can be described by only one parameter that is absolutely not a function of the attitude parameterization used.

Being the product of rotation matrices, matrix $\Delta = \boldsymbol{T}\boldsymbol{A}^T$ is a rotation matrix as well. This matrix represents the corrective rotation that, when applied to the estimation attitude, rotates it up to the true attitude, that is, $\Delta\boldsymbol{A} = \boldsymbol{T}\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{T}$. Being a rotation matrix, Δ has its own Euler axis and angle $(\boldsymbol{e}, \Phi)$, which implies $\Delta\boldsymbol{e} = \boldsymbol{e}$. Postmultiplying $\Delta = \boldsymbol{T}\boldsymbol{A}^T$ by the Euler axis $\boldsymbol{e}$, the relationship $\boldsymbol{T}^T\boldsymbol{e} = \boldsymbol{A}^T\boldsymbol{e}$ is obtained. This equation implies that an observed direction identified by the unit vector $\boldsymbol{e}$, in both the true and the estimated orientation, has an equal corresponding reference direction. In other words, an observed direction identified by $\boldsymbol{e}$ is errorless for the estimation matrix $\boldsymbol{A}$, no matter which matrix $\boldsymbol{A}$ is chosen. On the contrary, the direction identified by the unit vector $\boldsymbol{w}$, angularly separated from the Euler axis $\boldsymbol{e}$ by the angle ϑ, is affected by the error ϵ defined by

$$\cos\varepsilon = \cos^2\vartheta(1 - \cos\Phi) + \cos\Phi \tag{41}$$

where

$$\cos\vartheta = \boldsymbol{e}^T\boldsymbol{w} = \boldsymbol{e}^T\Delta\boldsymbol{w}, \qquad \cos\varepsilon = \boldsymbol{w}^T\Delta\boldsymbol{w} \tag{42}$$

Equation (41) is derived from the isosceles spherical triangle shown in Fig. 4, whose vertices are defined by $\boldsymbol{e}$, $\boldsymbol{w}$, and $\Delta\boldsymbol{w}$ unit vectors. The distribution of the rotational error ε, provided by Eq. (41), implies a maximum error of $\varepsilon = \Phi$ for directions perpendicular to the Euler axis ($\vartheta = \pi/2$) and a minimum error of $\varepsilon = 0$ for those directions parallel to the Euler axis, that is, at $\vartheta = 0$ and π. Thus, because the rotation error has to satisfy the assigned shape provided by Eq. (41), what characterizes it is its maximum value. This value, which is the Euler angle Φ of the matrix Δ, is also the maximum error obtained in the direction definition. This parameter identifies how far the matrix $\boldsymbol{A}$ is displaced from $\boldsymbol{T}$ and, therefore, describes the attitude error. The Euler angle Φ of the rotation matrix Δ is given by

$$\cos\Phi = \frac{\mathrm{tr}[\Delta] - 1}{2} = \frac{\mathrm{tr}[\boldsymbol{T}\boldsymbol{A}^T] - 1}{2} \tag{43}$$

This equation is easily demonstrated because the trace of a matrix is an invariant with respect to the similar transformations, and therefore it is equal to the sum of the eigenvalues, that is,

$$\mathrm{tr}[\boldsymbol{T}\boldsymbol{A}^T] = \sum_{i=1-3} \lambda_i = \lambda_1 + \lambda_2 + \lambda_3$$

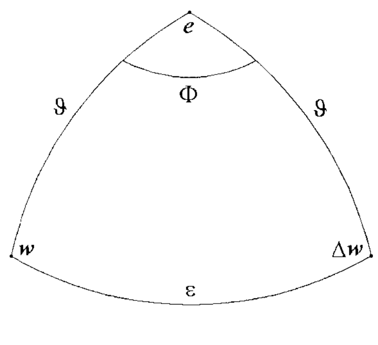

Fig. 4 Attitude error geometry.

and the eigenvalues of a rotation matrix are $\lambda_1 = 1$, $\lambda_2 = \cos\Phi + i\sin\Phi$, and $\lambda_3 = \cos\Phi - i\sin\Phi$. It follows that $\mathrm{tr}[\boldsymbol{T}\boldsymbol{A}^T] = 1 + 2\cos\Phi$. Consequently, the $\boldsymbol{A}$ matrices farthest from the $\boldsymbol{T}$ matrix imply $\Phi = \pi$, that is, $\mathrm{tr}[\boldsymbol{T}\boldsymbol{A}^T] = 1 + 2\cos\Phi = -1$, whereas all of the closest entail $\Phi = 0$, for which $\mathrm{tr}[\boldsymbol{T}\boldsymbol{A}^T] = 3$.

Unlike displacement, which allows an open error range (up to infinite), the rotation error is limited to a closed set and, furthermore, has a predefined shape that implies a zero error along one direction and a maximum error in the direction perpendicular to it. Based on this rotation error definition, the next section, which is dedicated to the accuracy tests, compares the optimal estimation matrices obtained from the Wahba criterion and the proposed new one with respect to the true attitude matrix.

Accuracy Tests

Accuracy tests have been performed on the basis of 1000 random attitude data sets and for n ranging from 2 to 10. Each data set has been produced as follows: 1) generation of a true attitude matrix $\boldsymbol{T}$ using Eq. (1) by a random choice of the Euler axis $\boldsymbol{e}$ and the Euler angle Φ; 2) generation of n random precision angles β_i for the three cases of $\beta_{\max} = 0.001$, 0.01, and 0.1 deg (where $0 \le \beta_i \le \beta_{\max}$); 3) generation of n random reference unit vectors $\boldsymbol{v}_i$; 4) evaluation of the n true observed unit vectors $\boldsymbol{S}_i = \boldsymbol{T}\boldsymbol{v}_i$; 5) evaluation of the n observed unit vectors $\boldsymbol{s}_i$, which are affected by a noise obtained by rotating the $\boldsymbol{S}_i$ by a random angle $\varphi_i (0 \le \varphi_i \le \beta_i, i = 1 - n)$ about a random axis perpendicular to $\boldsymbol{S}_i$; and 6) computation of the relative weights α_i and ξ_i according to Eq. (2) and Eqs. (11) and (12), respectively. For each attitude data set, which consists of the n data $[\xi_i, \alpha_i, \boldsymbol{v}_i, \boldsymbol{s}_i]$, the attitude estimation matrices $\boldsymbol{A}_W$ and $\boldsymbol{A}_M$ are obtained using the $\boldsymbol{q}$-Method algorithm (which fully complies with the Wahba cost function) and the Euler-$\boldsymbol{q}$ algorithm, respectively. Then, according to Eq. (43), the errors ε_W and ε_M provided by

$$\begin{aligned} \varepsilon_W &= \cos^{-1}\left\{\tfrac{1}{2}\left(\mathrm{tr}\left[\boldsymbol{T}\boldsymbol{A}_W^T\right] - 1\right)\right\} \\ \varepsilon_M &= \cos^{-1}\left\{\tfrac{1}{2}\left(\mathrm{tr}\left[\boldsymbol{T}\boldsymbol{A}_W^T\right] - 1\right)\right\} \end{aligned} \tag{44}$$

are computed. Figure 5 plots the average values of the ε_W and ε_M errors obtained from the tests. This plot shows that the Wahba cost function is, on the average, better than the proposed one. However, the precision loss, indicated by the difference $\varepsilon_M - \varepsilon_W$, decreases as n increases and decreases as the sensor errors β_i decrease. For instance, in the worst case of $n = 2$, the average of the maximum attitude errors are 1) $\varepsilon_W = 0.07$ deg and $\varepsilon_M = 0.08$ deg

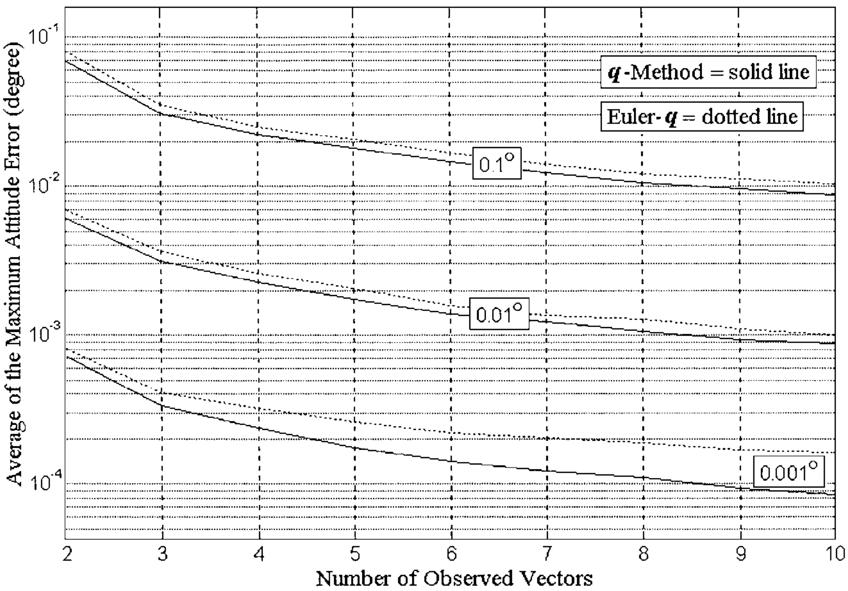

Fig. 5 Accuracy test results.

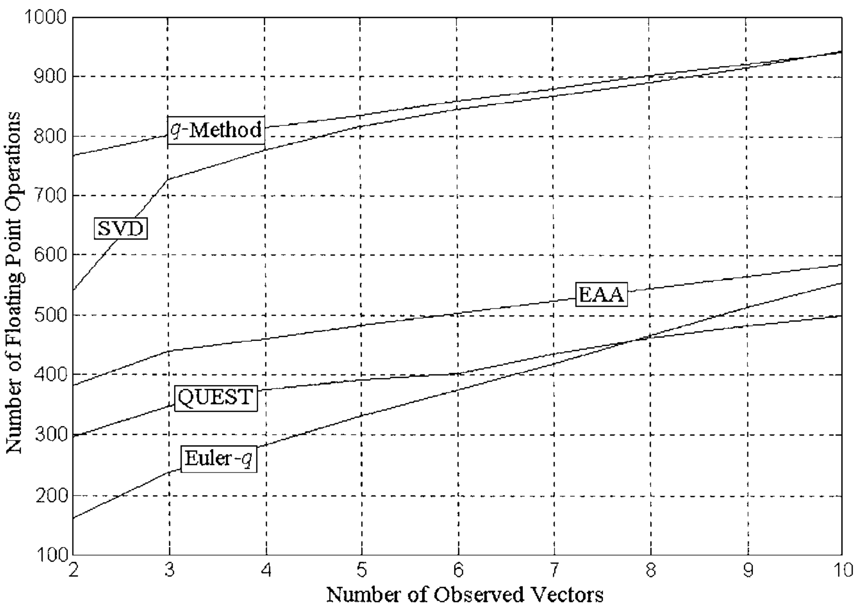

Fig. 6 Speed test results.

for $\beta_{\max} = 0.1$ deg (inaccurate sensors), 2) $\varepsilon_W = 0.006$ deg and $\varepsilon_M = 0.007$ deg for $\beta_{\max} = 0.01$ deg (accurate sensors), and 3) $\varepsilon_W = 0.0007$ deg and $\varepsilon_M = 0.0008$ deg for $\beta_{\max} = 0.001$ deg (very accurate sensors). We emphasize again that these values represent the maximum attitude errors in the worst case of $n = 2$. Hence, the Euler-$\boldsymbol{q}$ precision loss can be considered acceptable for many practical applications.

Speed Tests

As was done for accuracy tests, the speed tests also were performed using MATLAB[11] software on a Pentium personal computer. The index used to evaluate the algorithm computational speed is the MATLAB flops function, which evaluates the approximate cumulative number of floating point operations. Figure 6 shows the averages of the speed test indices obtained by 1000 tests, $n = 2$–10, and for $0 \leq \beta_i \leq \beta_{\max} = 0.1$ deg. Four different optimal attitude determination algorithms have been compared with Euler-$\boldsymbol{q}$ (tolerance $\zeta = 10^{-2}$; closed-form solutions for the λ_i, and no additional vectors used). These algorithms are 1) $\boldsymbol{q}$-Method,[2] 2) QUEST[3] (MSR tolerance $= 10^{-3}$), 3) SVD,[4] and 4) EAA.[6] Figure 6 shows Euler-$\boldsymbol{q}$ to be the fastest algorithm for the optimal attitude estimation when $n < 8$ observed vectors are available. This result confirms Euler-$\boldsymbol{q}$ as quite suitable for a fast optimal attitude estimation.

Conclusions

This paper presents a new cost function for computing the optimal Euler axis that is based on the eigenaxis rotational property. The mathematical procedure leads to a solution form that is similar to that of the $\boldsymbol{q}$-Method but that requires the eigenanalysis of a 3×3 matrix instead of a 4×4 one. The resulting new attitude estimation algorithm Euler-$\boldsymbol{q}$ is derived in a closed-form solution. The Euler-$\boldsymbol{q}$ singularity is avoided using the method of sequential rotations. Then, to compare the accuracy obtained by using the Wahba optimal definition with the proposed one, the rotation error is analyzed, and the parameter describing it is introduced. Numerical accuracy tests demonstrate that the new cost function describing the optimality criterion is a little less accurate than the Wahba criterion and that the precision loss decreases as the number of observed vectors increases and as the accuracy of the observed vectors increases. Speed tests show Euler-$\boldsymbol{q}$, for a number of observed vectors fewer than eight and with respect to the four most known attitude estimation algorithms, to be the fastest. This result demonstrates that Euler-$\boldsymbol{q}$ is suitable when a fast optimal attitude estimation is required, especially when many accurate observed vectors are available, e.g., wide-field-of-view star trackers.

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